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The nonhomogeneous elastodynamics problem*

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SUMMARY

The solutions of nonhomogeneous boundary value problems which arise from the study of the dynamics of bounded, elastic solids are represented by the superposition of two components: (a) a "quasi-static" solution which satisfies the nonhomogeneous boundary conditions, and (b) an eigenfunction expansion which satisfies the corresponding homogeneous boundary conditions. The method obviates the frequently used but often cumbersome technique of integral transforms, and a resolution of the problem is achieved by the use of classical mathematical analysis. The general theory developed is illustrated by considering the problem of point symmetric motion of a suddenly loaded spherical shell for which a complete solution is presented.

1. Introduction

One of the central problems of the classical, linear theory of elastodynamics is concerned with the determination of the displacement field and associated stress field in a bounded solid subjected to time and space dependent body forces and admissible boundary conditions, i.e., in the general case we have to solve a system of nonhomogeneous, partial differential equations with nonhomogeneous boundary conditions. In the case of homogeneous boundary conditions, solutions are often represented by an eigenfunction expansion, with each eigenfunction satisfying the imposed homogeneous boundary condition. In the case of nonhomogeneous boundary conditions, an eigenfunction expansion generally fails to satisfy the required boundary conditions, and the method of integral transforms has been successfully applied by many investigators to obtain a solution. Apart from the fact that each case requires individual judgement in the selection of the appropriate transform kernel, the problem of inversion is often quite difficult, and occasionally the required transform inverse does not exist.

In order to circumvent these difficulties, and to facilitate a solution by classical mathematical techniques alone, it is possible to adopt the following point of view. The (symbolic) solution u is assumed to be represented by the superposition of two parts $u=v+w$, where the "quasi-static" solution v satisfies the given problem with all inertia terms deleted. Thus v satisfies the reduced, nonhomogeneous field equations as well as the nonhomogeneous boundary conditions on u. Although v depends on the space variables (x^1, x^2, x^3) as well as on time t, time is a parameter in this part of the solution. The second part of the solution w is taken as an eigenfunction expansion such that the eigenfunctions satisfy certain homogeneous boundary conditions. The method proves to be useful for the solution of nonhomogeneous field equations, even if the boundary conditions on u are homogeneous. In this case both v and w are required to satisfy the same homogeneous boundary conditions, and the resulting solution representation is often found to have convergence characteristics which are superior to the conventional eigenfunction expansion of u.

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The concepts underlying the present approach are not new, and special, isolated applications to problems of the vibrating string and the Euler–Bernoulli beam can be found in the classical papers of Duhamel [1], and Phillips [2]. Recent generalizations of this technique and applications to the problems of vibrations of beams, plates, etc., can be found in references [3] through [8]. The concepts discussed above are generally applicable to linear partial differential equations (see [9] through [11]). Some of heuristic considerations underlying the present method were contributed by D. Williams in [12], although his work is restricted to one-dimensional structures and neglects to treat the case of time-dependent boundary conditions. We now proceed to develop the general method for the case of elastic bodies of bounded extent within the framework of the usual small deformation assumptions. In the following, general (curvilinear) coordinates are employed, and we adopt the notation of [13].

2. Statement and resolution of the boundary value problem

We consider a bounded, elastic solid which occupies the region $V+S$ in its undeformed configuration, where V is an open domain and S is its boundary. We wish to find the displacement field $u^i(x^1, x^2, x^3, t) \equiv u^i(x, t)$ which satisfies the stress-equations of motion

$$\tau^{ij}|_j + B^i = \rho \ddot{u}^i \tag{1}$$

throughout V for all $t>0$, the initial conditions

$$u^i(x, 0) = f^i(x), \quad \dot{u}^i(x, 0) = g^i(x) \tag{2}$$

throughout V at $t=0$, and the following admissible boundary conditions on S for all $t>0$:

(a) the displacement vector is specified on S_1, i.e.,

$$u^i = F^i(x, t) \text{ on } S_1 \tag{3a}$$

(b) the traction vector is specified on S_2, i.e.,

$$\tau^{ij} n_j = G^i(x, t) \text{ on } S_2 \tag{3b}$$

where $\hat{n}$ is the unit (outer) normal vector relative to S.

(c) the normal displacement and shear stress are specified on S_3, i.e.,

$$u^i n_i = u_{(n)}(x, t), \quad \tau^{ij} n_j - \tau^{kl} n_k n_l n^i = \tau^i(x, t) \text{ on } S_3 \tag{3c}$$

(d) the normal stress and tangential displacement are specified on S_4, i.e.,

$$\tau^{ij} n_i n_j = \sigma(x, t), \quad u^i - u^j n_j n^i = u^i_{(t)}(x, t) \text{ on } S_4 . \tag{3d}$$

(e) the traction vector is specified to include a linear elastic restoring force $(-\alpha^{ij} u_j)$ on S_5, i.e.,

$$\tau^{ij} n_j + \alpha^{ij} u_j = H^i(x, t) \text{ on } S_5 \tag{3e}$$

where the elastic foundation moduli satisfy the conditions $\alpha^{ij}(x) = \alpha^{ji}(x)$ and $2W_\alpha = \alpha^{ij} u_i u_j > 0$ for $u_i \neq 0$ on S_5. None of the sets S_1 through S_5 need be connected, and in any particular case one or more of them may be empty, provided that $S = S_1 + S_2 + S_3 + S_4 + S_5$. The stress and strain tensors are related to the displacement vector through the familiar relations

$$2e_{ij} = u_i|_j + u_j|_i \tag{4}$$

$$\tau^{ij} = C^{ijkl} e_{kl} = C^{ijkl} u_k|_l \tag{5}$$

where

$$C^{ijkl} = C^{jikl} = C^{ijlk} = C^{klij} \tag{6}$$

and $2W_c = C^{ijkl} e_{ij} e_{kl} > 0$ for $e_{ij} \neq 0$ in V. If the density $\rho(x) > 0$, the body force vector $B^i(x, t)$ and the twenty-one independent elastic constants $C^{ijkl}(x)$ are specified in V, then one can show that the solution of the problem posed by equations (1) through (6) is unique. The uniqueness proof is based on the positive definite character of the potential energy densities W_α and W_c and of the kinetic energy density $\frac{1}{2}(\rho \dot{u}^i \dot{u}_i)$.

We now propose to characterize the solution of the problem posed by (1) through (6) by

$$u^i(x, t) = v^i(x, t) + \sum_s W^i_{(s)}(x) q_s(t) \tag{7}$$

where $v^i(x, t)$ is the "quasi-static" solution of the reduced problem, defined by (1), (3), (4), (5), and (6), when the inertia terms in (1) are deleted, i.e.,

$$\tau^{ij}(v)|_j + B^i = 0 \text{ in } V \tag{8a}$$

where

$$\tau^{ij}(v) = C^{ijkl} v_k|_l \tag{8b}$$

and

$$v^i = F^i \text{ on } S_1 \tag{9a}$$

$$\tau^{ij}(v) n_j = G^i \text{ on } S_2 \tag{9b}$$

$$v^i n_i = u_{(n)}, \quad \tau^{ij}(v) n_j - \tau^{kl}(v) n_k n_l n^i = \tau^i \text{ on } S_3 \tag{9c}$$

$$\tau^{ij}(v) n_i n_j = \sigma, \quad v^i - v^j n_j n^i = u^i_{(t)} \text{ on } S_4 \tag{9d}$$

$$\tau^{ij}(v) n_j + \alpha^{ij} v_j = H^i \text{ on } S_5 . \tag{9e}$$

The eigenfunctions $W^i_{(s)}(x)$ of the associated homogeneous problem are characterized by the equations

$$\tau^{ij}_{(s)}|_j + \rho\omega_s^2 W^i_{(s)} = 0 \text{ in } V \tag{10a}$$

where

$$\tau^{ij}_{(s)} = C^{ijkl} W^{(s)}_k|_l \tag{10b}$$

and the homogeneous form of the boundary conditions (3), i.e.,

$$W^i_{(s)} = 0 \text{ on } S_1 \tag{11a}$$

$$\tau^{ij}_{(s)} n_j = 0 \text{ on } S_2 \tag{11b}$$

$$W^i_{(s)} n_i = 0, \quad \tau^{ij}_{(s)} n_j - \tau^{kl}_{(s)} n_k n_l n^i = 0 \text{ on } S_3 \tag{11c}$$

$$\tau^{ij}_{(s)} n_i n_j = 0, \quad W^i_{(s)} - W^j_{(s)} n_j n^i = 0 \text{ on } S_4 \tag{11d}$$

$$\tau^{ij}_{(s)} n_j + \alpha^{ij} W^{(s)}_j = 0 \text{ on } S_5 . \tag{11e}$$

The trivial solution $W^i_{(s)} \equiv 0$ and the rigid body displacement field $\boldsymbol{W}_{(s)} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{R}_0$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors and $\boldsymbol{R}_0$ is the position vector in V, are excluded from the set of eigenfunctions. To each eigenfunction $W^i_{(s)}$ there corresponds an eigenvalue ω_s^2, where ω_s is a natural frequency of vibration. We shall assume that the characterization of the eigenvalue problem (10) and (11) results in a denumerable set of solutions $\{W^i_{(r)}, \omega_r;\ r = 1, 2, 3, \ldots\}$. The validity of this assumption is discussed in [14] and [15]. The eigenvalues can be shown to be both real and positive numbers (see pp. 180–181 of [16]). If the rigid body motion is not an admissible eigenfunction, the case of a vanishing eigenvalue does not arise, and the eigenvalues can be ordered as follows: $0 < \omega_1^2 \leqq \omega_2^2 \leqq \omega_3^2 \leqq \ldots$ If there are no degeneracies (i.e., if the eigenvalues are distinct), the eigenfunctions can be shown to be orthogonal (see p. 180 of [16]). If degeneracies occur, i.e., if two or more different eigenfunctions correspond to the same eigenvalue, each set of degenerate eigenfunctions can be orthogonalized by the Gram–Schmidt process. In either case we have

$$\langle \rho W^i_{(r)} W^{(s)}_i \rangle_V = \delta^s_r \tag{12}$$

where the symbol $\langle\ \rangle_V$ denotes the volume integral of the enclosed expression over the region V. Equation (12) also implies that the eigenfunctions $W^i_{(r)}$ have been suitably normalized.

In view of equations (3) and (9), the assumed solution (7) satisfies the non-homogeneous boundary conditions (3). To determine the scalar functions $q_s(t)$, we now substitute (7) into (1) and (5). If, in addition, we apply (8) and (10), we obtain

$$\sum_s \rho W^i_{(s)}(\ddot{q}_s+\omega_s^2 q_s) = -\rho\ddot{v}^i . \tag{13a}$$

Similarly, if we set $t=0$ in (7) and substitute the resulting equations into (2), we readily obtain

$$\sum_s W^i_{(s)} q_s(0) = f^i(x)-v^i(x,0) \tag{13b}$$

$$\sum_s W^i_{(s)} \dot{q}_s(0) = g^i(x)-\dot{v}^i(x,0) . \tag{13c}$$

We now multiply (13a), (13b), and (13c) by $W_i^{(r)}$, $\rho W_i^{(r)}$, and $\rho W_i^{(r)}$, respectively, and integrate the resulting equations over the volume V. Upon application of (12), we obtain

$$\ddot{q}_r+\omega_r^2 q_r = \ddot{Q}_r(t) \text{ for } t>0 \tag{14a}$$

$$q_r(0)-Q_r(0) = \langle \rho W_i^{(r)} f^i\rangle_V \tag{14b}$$

$$\dot{q}_r(0)-\dot{Q}_r(0) = \langle \rho W_i^{(r)} g^i\rangle_V \tag{14c}$$

where

$$Q_r(t) = -\langle \rho v^i W_i^{(r)}\rangle_V \tag{15}$$

A more convenient form of $Q_r(t)$ can be obtained with the aid of (10) as follows:

$$\omega_r^2 Q_r(t) = \langle -\rho\omega_r^2 W^i_{(r)} v_i\rangle_V = \langle \tau^{ij}_{(r)}|_j v_i\rangle_V .$$

However, in view of (5), (6), and the definition of the covariant derivative, we have

$$(\tau^{ij}_{(r)} v_i)|_j - \tau^{ij}_{(r)}|_j v_i = \tau^{ij}_{(r)} v_i|_j = \tau^{ij}(v) W_i^{(r)}|_j = (\tau^{ij}(v) W_i^{(r)})|_j - \tau^{ij}(v)|_j W_i^{(r)} .$$

This result, together with the integral theorem of Green/Gauss/Ostrogradskii yields

$$\omega_r^2 Q_r(t) = \langle \tau^{ij}_{(r)} n_j v_i - \tau^{ij}(v) n_j W_i^{(r)}\rangle_S + \langle \tau^{ij}(v)|_j W_i^{(r)}\rangle_V$$

where the symbol $\langle\ \rangle_S$ denotes the integral of the enclosed expression over the region S. Substitution of (11), (8), and (9) into the preceding result then yields the desired form of $Q(t)$:

$$\omega_r^2 Q_r(t) = -\langle B^i W_i^{(r)}\rangle_V + \langle \tau^{ij}_{(r)} n_j F_i\rangle_{S_1} - \langle G^i W_i^{(r)}\rangle_{S_2} + \langle u_{(n)} \tau^{ij}_{(r)} n_i n_j - W_i^{(r)} \tau^i\rangle_{S_3} + \langle u_{(t)i} \tau^{ij}_{(r)} n_j - \sigma W^i_{(r)} n_i\rangle_{S_4} - \langle H^i W_i^{(r)}\rangle_{S_5} . \tag{16}$$

The solution of (14a) can be written

$$q_r(t) = [q_r(0)-Q_r(0)] \cos\omega_r t + \frac{1}{\omega_r}[\dot{q}_r(0)-\dot{Q}_r(0)] \sin\omega_r t + Q_r(t) - \omega_r \int_0^t Q_r(\xi) \sin\omega_r(t-\xi) d\xi \tag{17}$$

Thus the formal solution of the nonhomogeneous problem posed by (1) through (6) is given by (7), where v^i satisfies (8) and (9), $\{W_i^{(r)}, \omega_r; r=1, 2, 3, \ldots\}$ is the complete solution set of (10) and (11) which also satisfies (12), and the scalar functions $q_r(t)$ are given by (17) in conjunction with (16), (14b), and (14c).

3. Forced motion of a spherical shell

We now proceed to apply the general method to obtain a solution for the point symmetric, forced motion problem of a spherical shell referred to spherical coordinates (r, θ, φ). In this section we shall use the physical components of vectors and tensors, in contradistinction to the previous section, where tensor components were used. We assume that the shell is composed of a homogeneous, isotropic, elastic material and neglect body forces. It is also assumed that initially (at $t<0$) the shell is at rest, in its reference configuration in static equilibrium, and free from all surface tractions. At $t=0$ the concentric, spherical shell boundaries at $r=R_\mathrm{I}$ and $r=R_0$, $R_\mathrm{I}<R_0$, are subjected to the suddenly applied normal (compressive) surface pressures

$-P_I$ and $-P_0$, respectively. This problem is characterized by the equations

$$\left.\begin{aligned} &\tau_{rr,r}+r^{-1}(2\tau_{rr}-\tau_{\theta\theta}-\tau_{\varphi\varphi}) = \rho u_{,tt} && (18a)\\ &\tau_{rr} = (\lambda+2\mu)u_{,r}+2\lambda r^{-1}u && (18b)\\ &\tau_{\theta\theta} = \tau_{\varphi\varphi} = \lambda u_{,r}+2(\lambda+\mu)r^{-1}u && (18c)\end{aligned}\right\} R_I < r < R_0$$

where u is the radial displacement component. The boundary conditions are

$$\tau_{rr}(R_I, t) = -P_I H(t) \quad (19a)$$

$$\tau_{rr}(R_0, t) = -P_0 H(t) \quad (19b)$$

where $H(t)$ is the Heaviside unit step function. The initial conditions are

$$u(r, 0) = u_{,t}(r, 0) = 0\,. \quad (20)$$

With reference to equation (7), we seek a solution of the problem posed by (18), (19), and (20) in the form

$$u(r, t) = v(r, t) + \sum_{n=1}^{\infty} W_n(r)\, q_n(t)\,. \quad (21)$$

The associated "quasi-static" problem (see (8) and (9)) is here characterized by

$$v_{,rr}+2r^{-1}v_{,r}-2r^{-2}v = 0\,, \quad R_I < r < R_0 \quad (22)$$

and the boundary conditions

$$\left.\begin{aligned} &(\lambda+2\mu)v_{,r}(R_I, t)+2\lambda R_I^{-1}v(R_I, t) = -P_I H(t) && (23a)\\ &(\lambda+2\mu)v_{,r}(R_0, t)+2\lambda R_0^{-1}v(R_0, t) = -P_0 H(t) && (23b)\end{aligned}\right\} t>0$$

where t is a parameter. The solution of (22) subject to (23) is well-known (see p. 142 of [16]):

$$\frac{\rho C_D^2}{P_I}\frac{v}{R_0} = \frac{\beta H(t)}{(1-\beta^3)}\left[\frac{(\beta^3-P_0/P_I)}{(3-4\gamma^2)}\frac{r}{R_I} + \frac{1}{4\gamma^2}\left(1-\frac{P_0}{P_I}\right)\frac{R_I^2}{r^2}\right] \quad (24a)$$

$$\frac{\tau_{rr}(v)}{P_I} = \frac{H(t)}{(1-\beta^3)}\left[\beta^3 - \frac{P_0}{P_I} - \left(1-\frac{P_0}{P_I}\right)\frac{R_I^3}{r^3}\right] \quad (24b)$$

$$\frac{\tau_{\varphi\varphi}(v)}{P_I} = \frac{H(t)}{(1-\beta^3)}\left[\beta^3 - \frac{P_0}{P_I} + \frac{1}{2}\left(1-\frac{P_0}{P_I}\right)\frac{R_I^3}{r^3}\right] \quad (24c)$$

where $0<\beta=R_I/R_0<1$, $\gamma^2=(1-2\nu)/2(1-\nu)$ and $C_D=[(\lambda+2\mu)/\rho]^{\frac{1}{2}}$ is the speed of dilatational waves.

The eigenfunctions and associated eigenvalues satisfy the equation

$$W_{,rr}+2r^{-1}W_{,r}-2r^{-2}W = -\omega^2 C_D^{-2}W\,. \quad (25)$$

In view of (23), for the present case we require the boundary conditions

$$(\lambda+2\mu)W_{,r}(R_I)+2\lambda R_I^{-1}W(R_I) = 0 \quad (26a)$$

$$(\lambda+2\mu)W_{,r}(R_0)+2\lambda R_0^{-1}W(R_0) = 0\,. \quad (26b)$$

The eigenfunctions and eigenvalues characterized by (25) and (26) are known (see p. 287 of [16]). Upon normalization by

$$\left(\int_{R_I}^{R_0}\rho W^2 r^2 dr\right)^{\frac{1}{2}} = 1$$

we obtain

$$W_n = A_n h_1(z_n) \quad (27a)$$

$$\tau_{rr}^{(n)} = \rho C_D\omega_n A_n[h_0(z_n)-(4\gamma^2/z_n)h_1(z_n)] \quad (27b)$$

$$\tau_{\theta\theta}^{(n)} = \tau_{\varphi\varphi}^{(n)} = \rho C_D\omega_n A_n[(1-2\gamma^2)h_0(z_n)+(2\gamma^2/z_n)h_1(z_n)] \quad (27c)$$

where

$$A_n = (2\omega_n/\rho C_D R_0^2 I_n)^{\frac{1}{2}}, \quad I_n = \beta^2 b_n h_1^2(a_n) D_n$$
$$D_n = \alpha_n^2 [1+4\gamma^2(4\gamma^2-3)/b_n^2] - \beta[1+4\gamma^2(4\gamma^2-3)/a_n^2]$$
$$\alpha_n = \frac{(1-4\gamma^2/a_n^2)\cos a_n - (4\gamma^2/a_n)\sin a_n}{(1-4\gamma^2/b_n^2)\cos b_n - (4\gamma^2/b_n)\sin b_n}$$
$$z_n = r\omega_n/C_D, \quad a_n = R_I\omega_n/C_D, \quad b_n = R_0\omega_n/C_D$$
$$h_0(z_n) \equiv j_0(z_n) - (B_n/A_n) y_0(z_n), \quad h_1(z_n) \equiv j_1(z_n) - (B_n/A_n) y_1(z_n)$$

and where $j_0(z) = z^{-1} \sin z$ and $y_0(z) = -z^{-1} \cos z$ are spherical Bessel functions of the first and second kind, respectively, of order zero. The functions $j_1(z) = z^{-2} \sin z - z^{-1} \cos z$ and $y_1(z) = -z^{-2} \cos z - z^{-1} \sin z$ are spherical Bessel functions of the first and second kind, respectively, of order one, and

$$B_n/A_n = \frac{j_0(a_n) - (4\gamma^2/a_n) j_1(a_n)}{y_0(a_n) - (4\gamma^2/a_n) y_1(a_n)}.$$

The natural frequencies are roots of the transcendental equation

$$\tan \Omega = \varphi(\Omega) \tag{28}$$

where

$$\varphi(\Omega) = \frac{4\gamma^2(1-\beta)^2[\beta\Omega^2 + 4\gamma^2(1-\beta)^2]\Omega}{\beta^2\Omega^4 - 4\gamma^2(1-\beta)^2(1+\beta^2-4\gamma^2\beta)\Omega^2 + 16\gamma^4(1-\beta)^4}$$

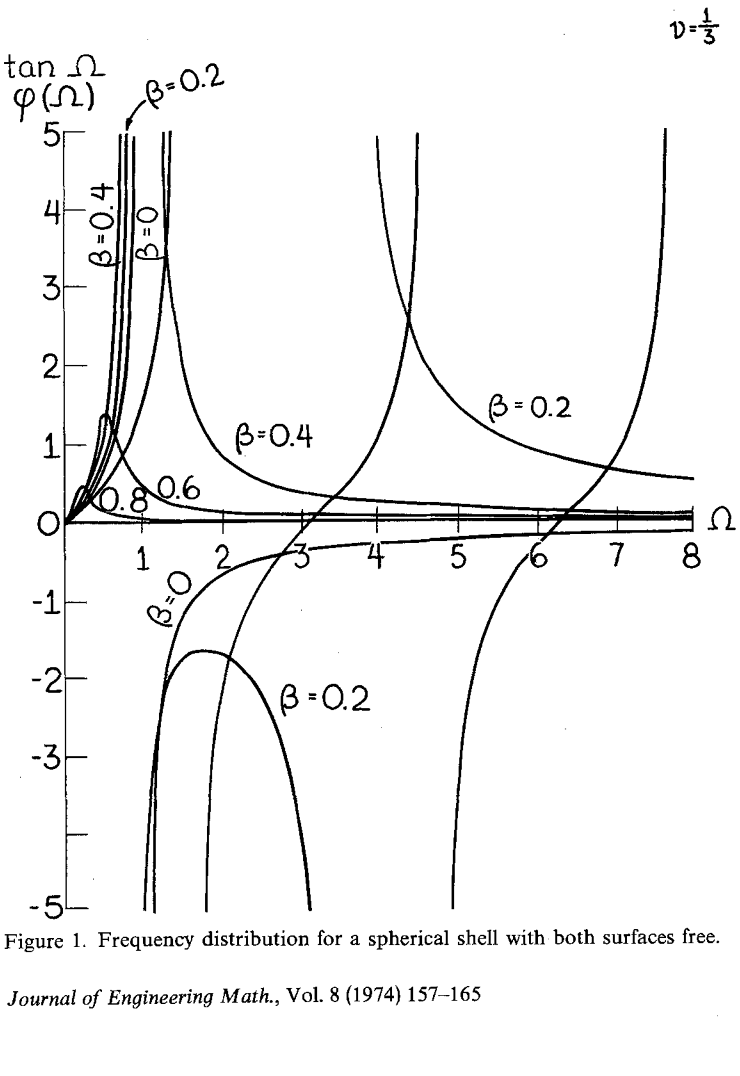

Figure 1. Frequency distribution for a spherical shell with both surfaces free.

and $\Omega = (R_0 - R_I)\omega/C_D$. Fig. 1 shows plots of tan Ω *vs.* Ω and $\varphi(\Omega)$ *vs.* Ω for various values of the radius ratio β and $\nu = \frac{1}{3}$. Each intersection of these curves is a solution of (28) and thus corresponds to a natural frequency. It is clear that for $0 < \beta < 1$, the frequency equation (28) has a denumerable infinity of distinct, positive, real roots.

The functions q_n satisfy equations (14a), with (see (16))

$$Q_n(t) = \omega_n^{-2}\left[R_0^2 P_0 W_n(R_0) - R_I^2 P_I W_n(R_I)\right] H(t) \tag{29}$$

and in view of (14) and (20),

$$q_n(0) = Q_n(0)\,, \quad \dot{q}_n = \dot{Q}_n(0)\,. \tag{30}$$

Upon substitution of (29) and (30) into (17), we obtain

$$q_n(t) = -R_I^2 P_I A_n h_1(a_n)\left(1 - \frac{\alpha_n}{\beta}\frac{P_0}{P_I}\right)\frac{\cos\omega_n t}{\omega_n^2}\,. \tag{31}$$

We now have all the components of the complete solution, which can be written as

$$\frac{\rho C_D^2}{P_I}\frac{u}{R_0} = \frac{\rho C_D^2}{P_I}\frac{v}{R_0} - 2(1-\beta)^2 \sum_{n=1}^{\infty}\left(1 - \frac{\alpha_n}{\beta}\frac{P_0}{P_I}\right)\frac{h_1(z_n)}{h_1(a_n)}\cdot\frac{\cos\Omega_n\tau}{\Omega_n^2 D_n} \tag{32a}$$

$$\frac{\tau_{rr}}{P_I} = \frac{\tau_{rr}(v)}{P_I} - 2(1-\beta)\sum_{n=1}^{\infty}\left(1 - \frac{\alpha_n}{\beta}\frac{P_0}{P_I}\right)\frac{h_0(z_n) - (4\gamma^2/z_n)h_1(z_n)}{h_1(a_n)}\cdot\frac{\cos\Omega_n\tau}{\Omega_n D_n} \tag{32b}$$

$$\frac{\tau_{\varphi\varphi}}{P_I} = \frac{\tau_{\varphi\varphi}(v)}{P_I} - 2(1-\beta)\sum_{n=1}^{\infty}\left(1 - \frac{\alpha_n}{\beta}\frac{P_0}{P_I}\right)\frac{(1-2\gamma^2)h_0(z_n) + (2\gamma^2/z_n)h_1(z_n)}{h_1(a_n)}\cdot\frac{\cos\Omega_n\tau}{\Omega_n D_n} \tag{32c}$$

where $\tau = C_D t/R_0(1-\beta)$. We consider the case $P_0/P_I = 0$ and $\nu = \frac{1}{3}$, and with the aid of (32) the displacement u and stress $\tau_{\varphi\varphi}$ were calculated at $r = R_I$ and are shown in Figures 2 and 3, respectively, as a function of the dimensionless time τ. The sudden application of the pressure $-P_I$ to the internal surface of the shell at $\tau = 0$ produces a dilatational wave which propagates

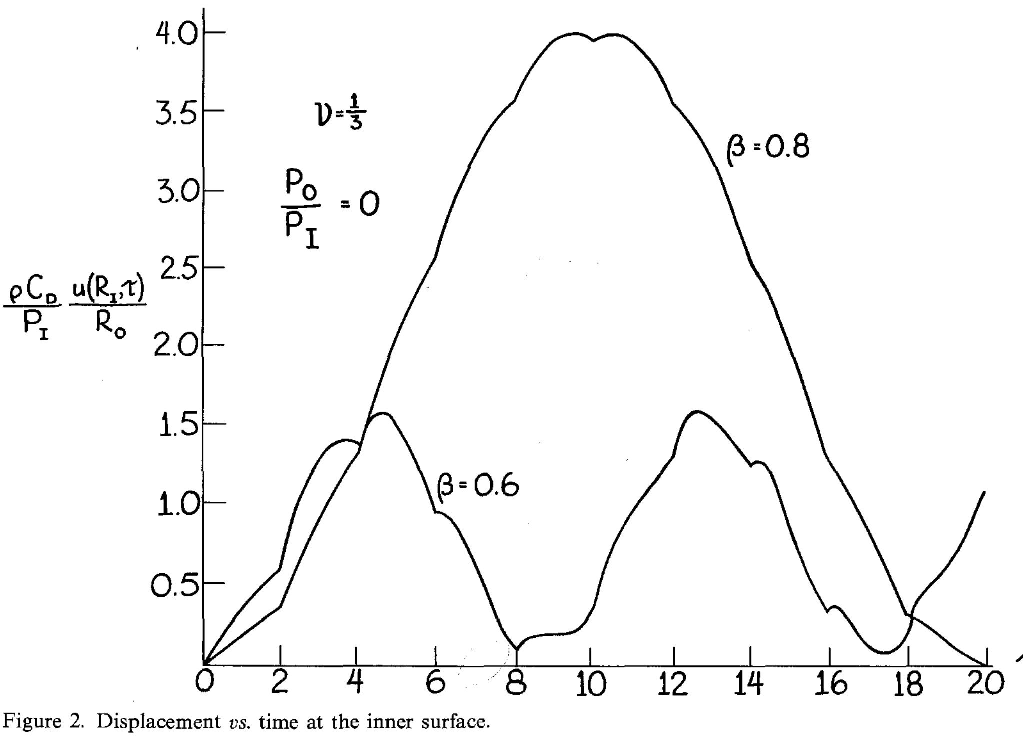

Figure 2. Displacement *vs.* time at the inner surface.

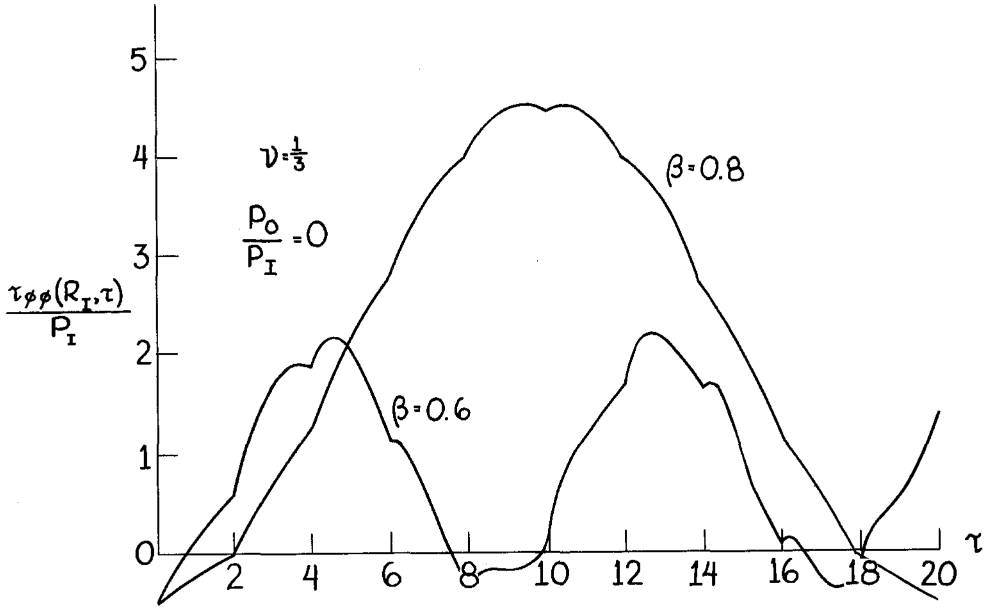

Figure 3. Stress *vs.* time at the inner surface.

through the shell to the outer surface from which it is reflected at $\tau = 1$. At $\tau = 2$ this reflected wave returns to the inner surface where it is again reflected, thus beginning another cycle. The arrival of these dilatational waves at the inner surface of the shell at the times $\tau = 2, 4, 6, 8$, etc., is clearly evident in Figures 2 and 3. Because of the point symmetric character of the present problem there is only one nonvanishing displacement component. For an application of the present method of solution to the case of three (generally) nonvanishing displacement components, the reader is referred to [17].

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